外 積 外 傳

$\boxed{\text{定義 1}}$：$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

$\boxed{\text{定義 2}}$：$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

$\boxed{\text{定義 3}}$：$\vec{a} = (a_1, a_2, a_3) = a_1 i + a_2 j + a_3 k$ ，其中$\begin{cases} i = (1,0,0) \\ j = (0,1,0) \\ k = (0,0,1) \end{cases}$

$\boxed{\text{定義 4}}$：$\vec{a} = (a_1, a_2, a_3)$ ，$\vec{b} = (b_1, b_2, b_3)$ ，則

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)(1,0,0) - (a_1b_3 - a_3b_1)(0,1,0) + (a_1b_2 - a_2b_1)(0,0,1)$$

$$= (a_2b_3 - a_3b_2, -(a_1b_3 - a_3b_1), a_1b_2 - a_2b_1)$$

$$= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

$\boxed{\text{定理 1}}$：$\vec{a} \times \vec{0} = \vec{0} \times \vec{a} = \vec{0}$

$\boxed{\text{定理 2}}$：$\vec{a}$與$\vec{b}$爲非零向量，則$\vec{a} \perp (\vec{a} \times \vec{b})$ 且 $\vec{b} \perp (\vec{a} \times \vec{b})$

$\boxed{\text{證明}}$：$\vec{a} \cdot (\vec{a} \times \vec{b}) = (a_1, a_2, a_3) \cdot (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b)$

$$= (a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1)$$

$$= (0,0,0)$$

$\boxed{\text{定理 3}}$：$\vec{a} \,//\, \vec{b}$ ，則$\vec{a} \times \vec{b} = \vec{0}$

$\boxed{\text{定理 4}}$：$\vec{a}$與$\vec{b}$爲非零向量，則$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\boxed{\text{定理 5}}$：$\vec{a}$與$\vec{b}$爲非零向量，其夾角爲θ ，則$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$\boxed{\text{證明}}$：$|\vec{a} \times \vec{b}|^2 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + (-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix})^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$

$= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$

$= (a_2{}^2b_3{}^2 - 2a_2a_3b_2b_3 + a_3{}^2b_2{}^2 + a_3{}^2b_1{}^2 - 2a_1a_3b_1b_3 + a_1{}^2b_3{}^2 + a_1{}^2b_2{}^2 - 2a_1a_2b_1b_2 + a_2{}^2b_1{}^2)$

$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$

$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

$= |\vec{a}|^2|\vec{b}|^2 - (|\vec{a}| \cdot |\vec{b}|\cos\theta)^2$

$= |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta)$

$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta$

所以$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$\boxed{\text{定理 6}}$：$\vec{a}$ 、$\vec{b}$與$\vec{c}$爲非零向量，則$(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$ 且

$|(\vec{a} \times \vec{b}) \cdot \vec{c}| = |\vec{a} \cdot (\vec{b} \times \vec{c})|$ 爲由$\vec{a}$ 、$\vec{b}$與$\vec{c}$ 向量拓展成的平行六面體體積