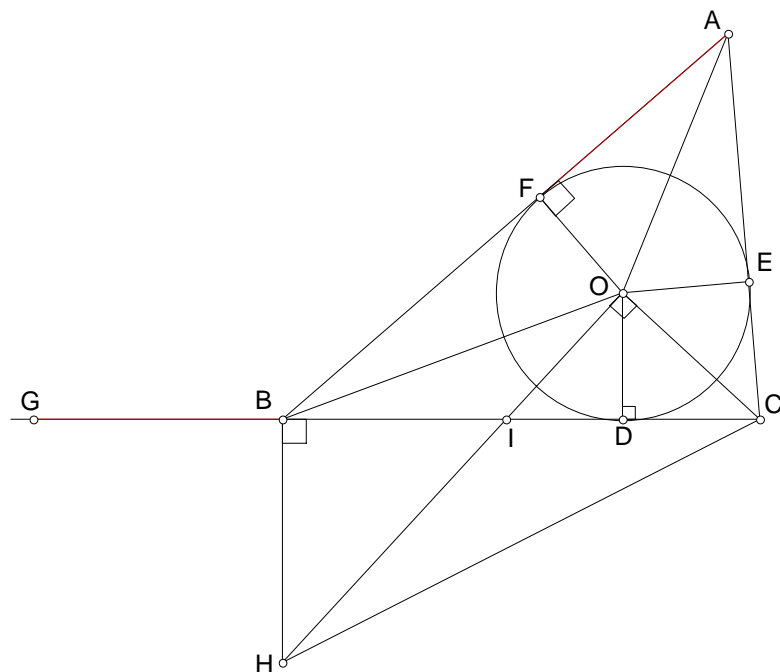


三角形 ABC ， $\overline{AB} = c, \overline{BC} = a, \overline{CA} = b$ 其面積為 $\sqrt{S(S-a)(S-b)(S-c)}$ ，其中

$$S = \frac{1}{2}(a+b+c)$$



證明：

1. 如圖，在 \overline{CB} 上取一點 G ，使得 $\overline{BG} = \overline{AF}$
2. 過 B 作 \overline{BC} 的垂線，過 O 作 \overline{OC} 的垂線，兩線交於 H
3. $BOCH$ 四點共圓（都在以 \overline{HC} 為直徑的圓上）， $\angle BHC + \angle BOC = 180^\circ$
4. 又因 $\angle FOB + \angle BOC = 180^\circ$ ，所以 $\angle BHC = \angle FOA$ ，所以 $\Delta AFO \sim \Delta CBH$ ，
 $\frac{\overline{BC}}{\overline{BH}} = \frac{\overline{AF}}{\overline{FO}} = \frac{\overline{BG}}{\overline{OD}}$ 即 $\frac{\overline{BC}}{\overline{BG}} = \frac{\overline{BH}}{\overline{OD}}$ ，則 $\frac{\overline{BC} + \overline{BG}}{\overline{BG}} = \frac{\overline{BH} + \overline{OD}}{\overline{OD}}$
5. 因 $\Delta BIH \sim \Delta DIO$ ，所以 $\frac{\overline{BH}}{\overline{OD}} = \frac{\overline{BI}}{\overline{ID}}$ ，則 $\frac{\overline{BH} + \overline{OD}}{\overline{OD}} = \frac{\overline{BI} + \overline{ID}}{\overline{ID}} = \frac{\overline{BD}}{\overline{ID}}$
6. ΔOIC 中， $\overline{ID} \times \overline{DC} = \overline{OD}^2$
7. 由 4,5,6 可得 $\frac{\overline{BC} + \overline{BG}}{\overline{BG}} = \frac{\overline{GC}}{\overline{BG}} = \frac{\overline{BH} + \overline{OD}}{\overline{OD}} = \frac{\overline{BD}}{\overline{ID}} = \frac{\overline{BD} \times \overline{DC}}{\overline{ID} \times \overline{DC}} = \frac{\overline{BD} \times \overline{DC}}{\overline{OD}^2}$ ，則

$$\frac{\overline{GC}}{\overline{BG}} \times \frac{\overline{GC}}{\overline{GC}} = \frac{\overline{BD} \times \overline{DC}}{\overline{OD}^2}，(\overline{GC} \times \overline{OD})^2 = \overline{GC} \times \overline{BG} \times \overline{BD} \times \overline{DC}，再令 $\overline{OD} = r$ ，即$$

$$\left(\frac{a+b+c}{2} \times r\right)^2 = \left(\frac{a+b+c}{2}\right) \times \left(\frac{a+b+c-2a}{2}\right) \times \left(\frac{a+b+c-2b}{2}\right) \times \left(\frac{a+b+c-2c}{2}\right)$$

$$\text{所以 } \triangle ABC \text{ 的面積} = \sqrt{S(S-a)(S-b)(S-c)}$$

另證：

$\triangle ABC$ 的面積 =

$$\begin{aligned} \frac{1}{2}ab \sin C &= \frac{1}{2}ab \sqrt{1 - \cos^2 C} = \sqrt{\left(\frac{ab}{2}\right)^2 - \left(\frac{ab}{2}\right)^2 \times \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2} = \\ &= \sqrt{\left(\frac{ab}{2}\right)^2 - \left(\frac{a^2 + b^2 - c^2}{4}\right)^2} = \sqrt{\left(\frac{ab}{2} + \frac{a^2 + b^2 - c^2}{4}\right)\left(\frac{ab}{2} - \frac{a^2 + b^2 - c^2}{4}\right)} = \\ &= \sqrt{\left(\frac{(a+b)^2 - c^2}{4}\right)\left(\frac{c^2 - (a-b)^2}{4}\right)} = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{c+a-b}{2}\right)\left(\frac{c-a+b}{2}\right)} \\ &= \sqrt{S(S-a)(S-b)(S-c)} \end{aligned}$$